Asymptotic results and statistical procedures for time-changed Lévy processes sampled at hitting times

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Abstract

We provide asymptotic results and develop high frequency statistical procedures for time-changed Lévy processes sampled at random instants. The sampling times are given by first hitting times of symmetric barriers whose distance with respect to the starting point is equal to ε. This setting can be seen as a first step towards a model for tick-by-tick financial data allowing for large jumps. For a wide class of Lévy processes, we introduce a renormalization depending on ε, under which the Lévy process converges in law to an α-stable process as ε goes to 0. The convergence is extended to moments of hitting times and overshoots. In particular, these results allow us to construct consistent estimators of the time change and of the Blumenthal-Getoor index of the underlying Lévy process. Convergence rates and a central limit theorem are established under additional assumptions.

Key words: time-changed Lévy processes, statistics of high frequency data, stable processes, hitting times, overshoots, Blumenthal-Getoor index, central limit theorem

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1   Introduction

In the recent years, a large number of papers has been devoted to asymptotic results and statistical procedures for time-changed Lévy processes [15, 16, 39] and more general semimartingales [1, 4, 3, 2, 21], under high-frequency discrete sampling. The classical high frequency setting consists in observing n values of the process over a fixed time interval $[0, T]$ at deterministic sampling times $0 = t_0^n < t_1^n < \ldots < t_n^n = T$. Usually, asymptotic results are given as n goes to infinity and $\sup\{t_{i+1}^n - t_i^n\}$ goes to zero. Motivated by financial applications,

many papers focus more specifically on the asymptotic behavior of volatility estimators. For example, power variation estimators which are robust to jumps are studied in [7] and [29]. Since financial data are often seen as noisy observations of a semimartingale, limit theorems for volatility estimators under various kinds of perturbations have also been widely studied, mostly in the case of continuous semimartingales, see among others [6, 22, 35, 40].

In this paper we focus on time-changed Lévy models, that is, we assume that the process of interest Y is given by $Y_t = X_{S_t}$ where X is a one-dimensional Lévy process and S is a continuous increasing process (a time change), which plays the role of the integrated volatility in this setting. Time changed Lévy models were introduced into financial literature in [11] and their estimation from high frequency data with deterministic sampling was recently addressed in [15, 16].

In the context of ultra high-frequency financial data, the assumption of deterministic sampling times is arguably too restrictive. Several authors have therefore considered volatility estimation with endogenous sampling times [18, 20, 27, 32] but so far only in the context of continuous processes.

In this work we assume that the sampling times are given by first hitting times of symmetric barriers whose distance with respect to the starting point is equal to ε. More precisely, the process Y is observed at times $(T_i^\varepsilon)_{i \geq 0}$ with $T_0^\varepsilon = 0$ and $T_{i+1}^\varepsilon = \inf\{t > T_i^\varepsilon : |Y_t - Y_{T_i^\varepsilon}| \geq \varepsilon\}$ for $i \geq 1$. The parameter ε is the parameter driving the asymptotic and thus we will assume that ε goes to zero.

This scheme is probably the most simple and common endogenous sampling scheme. Moreover, in the spirit of [32] it can be seen as a first step towards a model for ultra high frequency financial data including jump effects. For example, Y could represent the unobservable efficient price process and $[-\varepsilon, \varepsilon]$ the bid-ask interval. However, a detailed financial interpretation of our model is left for further research. For practical application, the model should in particular be modified so that the observed values remain on the tick grid.

Our asymptotic results may more generally open the way for studying hedging and portfolio strategies with random endogenous readjustment dates (see e.g. [17, 33] for relevant examples in the setting of continuous processes) and for approximating the solutions of stochastic differential equations by Euler-type schemes with random discretization dates (see e.g., [24, 37]).

We focus on the class of Lévy processes such that for a suitable α, the rescaled process $(X_t^\varepsilon)_{t \geq 0} := (\varepsilon^{-1} X_{\varepsilon^\alpha t})_{t \geq 0}$ converges in law to an α-stable Lévy process X^* as ε goes to zero. This class turns out to be rather large, and contains in particular all Lévy processes with non-zero diffusion component, all finite variation Lévy processes with non-zero drift and also most parametric Lévy models found in the literature. We show that for such Lévy processes the moments of first exit times from intervals, and certain functionals of the overshoot converge

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to the corresponding functionals of the limiting stable process, which are often known explicitly.

These findings, which are of interest in their own right, allow us to prove limit theorems for quantities of the form

$$V^\varepsilon(f)_t = \sum_{T_i^\varepsilon \le t} f\big(\varepsilon^{-1}(Y_{T_i^\varepsilon} - Y_{T_{i-1}^\varepsilon})\big),$$

leading to consistent estimators of the time change and of the characteristics of X that are preserved by the limiting procedure, such as, for example, the Blumenthal-Getoor index of jump activity. In some cases, we are able to quantify the rate of convergence of the functionals of the rescaled process X^ε to the corresponding functionals of the limiting stable process X^*. From this, convergence rates and central limit theorems for our estimators can be deduced.

The paper is organized as follows. In Section 2, we study the convergence in law of the properly rescaled underlying Lévy process X as ε goes to zero. Asymptotic results for the first exit time and the overshoot (more precisely we study the value of the process at the first exit time which is directly related to the overshoot) are given in Section 3. The law of large numbers for $V^\varepsilon(f)$ is stated in Section 4, where we also discuss statistical applications. Finally, a multidimensional central limit theorem is given in Section 5. The proofs are relegated to Section 6.

2   Convergence of the rescaled process

In this section, we give results on the convergence in law of the properly rescaled process X as ε goes to zero. The convergences in law are given in the Skorohod space, for the usual Skorohod topology. These results will be essential for proving the law of large numbers and the central limit theorem. Let us first recall the definition of a strictly stable process and introduce other useful notation.

Preliminaries and notation   We denote by (A, ν, γ) the characteristic triplet of the one-dimensional Lévy process X, with respect to a truncation function h. This means that via the Lévy-Khintchine formula, the characteristic function of X_t is

$$E[e^{iuX_t}] = e^{t\psi(u)}, \quad \psi(u) = -\frac{Au^2}{2} + i\gamma u + \int_{\mathbb{R}} (e^{iux} - 1 - iuh(x))\nu(dx).$$

Unless otherwise specified, we assume $h(x) = -1 \vee x \wedge 1$.

A Lévy process X is called strictly α-stable for $\alpha \in (0, 2]$ if X_t has a strictly α-stable distribution for all t. This happens if and only if X is selfsimilar, that

is,
$$\forall a > 0, \quad \left(\frac{X_{at}}{a^{1/\alpha}}\right)_{t \geq 0} = (X_t)_{t \geq 0}, \text{ in law.}$$

As recalled in the following proposition, strictly stable Lévy processes can be described in terms of their characteristic triplet.

Proposition (Theorems 14.3, 14.7 in [38]). *Let X be a Lévy process with characteristic triplet (A, ν, γ).*

1. *X is strictly 2-stable if and only if $\nu = 0$ and $\gamma = 0$.*

2. *X is strictly α-stable with $1 < \alpha < 2$ if and only if $A = 0$, ν has a density of the form*
$$\nu(x) = \frac{c_+}{|x|^{1+\alpha}} 1_{x>0} + \frac{c_-}{|x|^{1+\alpha}} 1_{x<0}, \tag{1}$$
   and $\gamma_c = 0$ where $\gamma_c := \gamma - \int_{\mathbb{R}} (h(x) - x)\nu(dx)$ is the third component of the characteristic triplet of X with respect to the truncation function $h(x) = x$.

3. *X is strictly 1-stable if and only $A = 0$ and ν has a density of the form*
$$\nu(x) = \frac{c}{|x|^2}.$$

4. *X is strictly α-stable with $0 < \alpha < 1$ if and only if $A = 0$, ν has a density of the form (1) and $\gamma_0 = 0$, where $\gamma_0 := \gamma - \int_{\mathbb{R}} h(x)\nu(dx)$ is the third component of the characteristic triplet of X with respect to the truncation function $h = 0$.*

For $\alpha \in (0, 2]$ and $\varepsilon > 0$, we define the rescaled Lévy process X^ε via $X_t^\varepsilon := \varepsilon^{-1} X_{\varepsilon^\alpha t}$, $t \geq 0$. The first exit time by the rescaled process from the interval $(-1, 1)$ will be denoted by $\tau_1^\varepsilon := \inf\{t \geq 0 : |X_t^\varepsilon| \geq 1\}$. This time is directly related to the first exit time by the original process from the interval $(-\varepsilon, \varepsilon)$:
$$\inf\{t \geq 0 : |X_t| \geq \varepsilon\} = \varepsilon^\alpha \tau_1^\varepsilon.$$

Similarly, $X_{\tau_1^\varepsilon}^\varepsilon$ is equal to ε^{-1} times the value of X at first exit from $(-\varepsilon, \varepsilon)$. From the Lévy-Khintchine formula it is easy to see that the characteristic triplet $(A^\varepsilon, \nu^\varepsilon, \gamma^\varepsilon)$ of X^ε is given by
$$A^\varepsilon = A\varepsilon^{\alpha-2}; \tag{2}$$
$$\nu^\varepsilon(B) = \varepsilon^\alpha \nu(\{x : x/\varepsilon \in B\}), \quad B \in \mathcal{B}(\mathbb{R}); \tag{3}$$
$$\gamma^\varepsilon = \varepsilon^{\alpha-1}\{\gamma + \int_{\mathbb{R}} \nu(dx)(\varepsilon h(x/\varepsilon) - h(x))\}. \tag{4}$$

Assumptions To be able to prove the convergence of the properly rescaled process, we introduce two assumptions on the Lévy measure which will sometimes be imposed in the sequel:

($\mathbf{H}$-α) The Lévy measure ν has a density $\nu(x) = \frac{g(x)}{|x|^{1+\alpha}}$, where g is a nonnegative measurable function admitting left and right limits at zero:

$$c^+ := \lim_{x \downarrow 0} g(x), \quad c^- := \lim_{x \uparrow 0} g(x),$$

with $c_+ + c_- > 0$.

($\mathbf{H}'$-α) The Lévy measure ν satisfies ($\mathbf{H}$-α) and additionnally $c_+ c_- > 0$ and the function g is left- and right-Hölder continuous at zero with exponent $\theta > \alpha/2$:

$$\limsup_{x \downarrow 0} \frac{|g(x) - c_+|}{|x|^\theta} < \infty \quad \text{and} \quad \limsup_{x \uparrow 0} \frac{|g(x) - c_-|}{|x|^\theta} < \infty.$$

Convergence in law of the rescaled process We now establish a set of alternative conditions under which the rescaled process X^ε converges in law to a strictly stable process as $\varepsilon \to 0$. In the sequel, we will always work under one of these alternative assumptions. The following proposition, therefore, also serves as the definition of the limiting process X^* and of the scaling parameter α depending on the characteristics of X.

Proposition 1.

1. Let $\alpha = 2$ and $A > 0$. Then the process X^ε converges in law to a Lévy process X^* with characteristic triplet $(A, 0, 0)$, that is, to a Brownian motion with variance A at time $t = 1$.

2. Let $\alpha = 1$ and assume that X has finite variation (that is, $A = 0$ and $\int_{|x| \leq 1} |x| \nu(dx) < \infty$) and nonzero drift: $\gamma_0 := \gamma - \int_{\mathbb{R}} h(x) \nu(dx) \neq 0$. Then the process X^ε converges in law to the (deterministic) Lévy process X^* with characteristic triplet $(0, 0, \gamma_0)$.

3. Let $1 < \alpha < 2$ and assume that $A = 0$ and that the Lévy measure ν satisfies the condition ($\mathbf{H}$-α). Then the process X^ε converges in law to a strictly α-stable Lévy process X^* with Lévy density

$$\nu^*(x) = \frac{c_+ \mathbf{1}_{x>0} + c_- \mathbf{1}_{x<0}}{|x|^{1+\alpha}}. \tag{5}$$

4. Let $\alpha = 1$ and assume that $A = 0$ and that the Lévy measure ν satisfies the condition ($\mathbf{H}$-α) with $c^+ = c^- := c$ and with the function g satisfying

$$\int_0^1 \frac{|g(x) - g(-x)| dx}{x} < \infty.$$

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Then the process X^ε converges in law to a Lévy process X^* with characteristic triplet $(0, \nu^*, \gamma^*)$, where $\gamma^* = \gamma - \int_0^\infty \frac{g(x) - g(-x)}{x^2} h(x) dx$ and ν^* has Lévy density

$$\nu^*(x) = \frac{c}{|x|^2},$$

that is, to a strictly 1-stable Lévy process.

5. *Let $0 < \alpha < 1$ and assume that $A = 0$, the process has zero drift: $\gamma - \int_\mathbb{R} h(x)\nu(dx) = 0$ and that the Lévy measure ν satisfies the condition (**H-α**). Then the process X^ε converges in law to a strictly α-stable Lévy process X^* with Lévy density (5).*

Remark 1. This result is closely related to the convergence of tempered stable processes to stable processes studied in [36]. More precisely, in Theorem 3.1 of [36], Rosiński proves the results of parts 3, 4 and 5 under the additional assumption that the function g is completely monotone (but in the multidimensional setting).

Remark 2. The different alternative cases contain the main parametric models found in finance literature. We list several examples below.

- All models with a nonzero diffusion component (e.g., the models of Merton [30] and Kou [25]) satisfy Condition 1.

- The variance gamma model [28] with nonzero drift satisfies Condition 2.

- The normal inverse gaussian process (NIG), see [5], satisfies Condition 4. This can be seen directly from the form of the Lévy density

$$\nu(x) = \frac{C}{|x|} e^{Ax} K_1(B|x|),$$

where A, B and C are constants and K_1 is the modified Bessel function of the second kind, which satisfies $K_1(x) \sim \frac{1}{x}$ for $x \downarrow 0$.

- The CGMY process, see [10], that is, a Lévy process with no diffusion component and a Lévy density of the form

$$\nu(x) = \frac{C e^{-\lambda_- |x|}}{|x|^{1+\alpha}} 1_{x<0} + \frac{C e^{-\lambda_+ |x|}}{|x|^{1+\alpha}} 1_{x>0}, \tag{6}$$

satisfies Condition 3 if $1 < \alpha < 2$, Condition 4 if $\alpha = 1$, Condition 2 if $\alpha < 1$ and the process has nonzero drift, and Condition 5 if $\alpha < 1$ and the drift is zero.

3 Asymptotic results for the first exit time and the overshoot of Lévy processes out of small intervals

In this section, our aim is to study the first exit time and the overshoot corresponding to the exit of X from the interval $(-\varepsilon, \varepsilon)$. In order to work with quantities of order 1, we formulate our results in terms of τ_1^ε and $X_{\tau_1^\varepsilon}^\varepsilon$.

Convergence for the first exit time and overshoot We define τ^* as the first exit time by the limiting process X^* from the interval $(-1, 1)$. Observe that τ^* admits moments of any order. When X^* is a nontrivial α-stable process with $0 < \alpha < 2$, τ^* is dominated by the time of the first jump of X^* greater than 2 in absolute value, which has exponential distribution. In the case $\alpha = 2$ (Brownian motion) this is a classical result, see for example [12, 13].

Proposition 2. *Let X be a Lévy process satisfying one of the conditions 1–5 of Proposition 1 and let f be a bounded continuous function on $\mathbb{R}$. Then*

1. *$(\tau_1^\varepsilon, X_{\tau_1^\varepsilon}^\varepsilon)$ converges in law to $(\tau_1^*, X_{\tau_1^*}^*)$ as $\varepsilon \downarrow 0$.*

2. *$\lim_{\varepsilon \downarrow 0} E[(\tau_1^\varepsilon)^k f(X_{\tau_1^\varepsilon}^\varepsilon)] = E[(\tau_1^*)^k f(X_{\tau_1^*}^*)]$ for all $k \geq 1$.*

Remark 3. The weak convergence of the $X_{\tau_1^\varepsilon}^\varepsilon$ under Conditions 1 or 2 of Proposition 1 (actually in these two cases $|X_{\tau_1^\varepsilon}^\varepsilon| \to 1$) is a known result [14]. See also [26, Theorem 5.16] for a related result in the context of subordinators.

Remark 4. The moments of the exit time and the law of the overshoot for the limiting strictly stable process are often known explicitly.

- Under Condition 1 of Proposition 1, the limiting process is a Brownian motion, so $X_{\tau_1^*}^*$ equals 1 or -1 with probability $\frac{1}{2}$ and the law of τ_1^* is well known (see e.g., exercise II.3.10 in [31]).

- Under Condition 2 of Proposition 1, the limiting process is deterministic, so $\tau_1^* = \frac{1}{|\gamma_0|}$ and $X_{\tau_1^*}^* = \operatorname{sgn} \gamma_0$.

- Under Conditions 3–5, the first and second moments of the hitting time τ_1^* are given in [19] for the symmetric case, and the law of the overshoot is computed in [9] for the symmetric case and in [34] for the general case.

Rates of convergence for the first exit times and overshoots We now compute the rates of convergence of $E[\tau_1^\varepsilon]$ to $E[\tau_1^*]$ and of $E[f(X_{\tau_1^\varepsilon}^\varepsilon)]$ to $E[f(X_{\tau_1^*}^*)]$. These results either guarantee the asymptotic normality of the estimators provided in Section 4 or allow to establish a convergence rate or an error bound for these estimators in the cases when the bias asymptotically dominates the variance.

Proposition 3.

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1. Let X be a Lévy process satisfying Condition 1 of Proposition 1 such that its Lévy measure ν satisfies $\int_{|x|\leq 1} |x| \nu(dx) < \infty$ and let f be a bounded Lipschitz function on $\mathbb{R}$ with $f(-1) = f(1)$. Then

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-1}(E[\tau_1^\varepsilon] - E[\tau_1^*]) = 0 \quad and \quad \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}(E[f(X_{\tau_1^\varepsilon}^\varepsilon)] - E[f(X_{\tau_1^*}^*)]) = 0.$$

2. Let X be a Lévy process satisfying Condition 2 of Proposition 1 such that its Lévy measure ν satisfies $\int_{|x|\leq 1} |x|^\beta \nu(dx)$ for some $\beta \in (0, 1)$, and let f be a bounded Lipschitz function on $\mathbb{R}$. Then

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-(1-\beta-\delta)}(E[\tau_1^\varepsilon] - E[\tau_1^*]) = 0 \tag{7}$$

and

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-(1-\beta-\delta)}(E[f(X_{\tau_1^\varepsilon}^\varepsilon)] - E[f(X_{\tau_1^*}^*)]) = 0 \tag{8}$$

for all $\delta > 0$.

3. Let X be a Lévy process satisfying Condition 3 of Proposition 1, Assumption $(\mathbf{H}'\text{-}\alpha)$ and the condition

$$\gamma = \int_{\mathbb{R}} \left(h(x)\frac{d\nu}{d\nu^*}(x) - x \right) \nu^*(dx). \tag{9}$$

or

let X be a Lévy process satisfying Condition 4 of Proposition 1 and Assumption $(\mathbf{H}'\text{-}\alpha)$

or

let X be a Lévy process satisfying Condition 5 of Proposition 1 and Assumption $(\mathbf{H}'\text{-}\alpha)$. Let f be a bounded continuous function on $\mathbb{R}$.

Then

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-\alpha/2}(E[\tau_1^\varepsilon] - E[\tau_1^*]) = 0 \quad and \quad \lim_{\varepsilon\downarrow 0} \varepsilon^{-\alpha/2}(E[f(X_{\tau_1^\varepsilon}^\varepsilon)] - E[f(X_{\tau_1^*}^*)]) = 0. \tag{10}$$

Remark 5. As we shall see below, Conditions 1 and 3 lead to a central limit theorem for the estimators constructed in the following sections, while Condition 2 provides a convergence rate without ensuring asymptotic normality. A natural question is what happens in the case where the Lévy process satisfies Condition 3 of Proposition 1 but the drift constraint (9) is not satisfied. In this case, we have been unable to obtain a convergence rate, due to unsufficient regularity of the functions of type $E^x[\tau_1^*]$ and $E^x[f(X_{\tau_1^*}^*)]$. However the following example shows that the estimate (10) may not hold in this case, and therefore one cannot hope to obtain a limit theorem without bias.

Let X be a Lévy process with characteristic triplet $(0, \nu, \gamma_c)$ with respect to the truncation function $h(x) = x$ and ν given by (1) with $c_+ = c_-$ and $1 < \alpha < 2$. Assume $\gamma_c > 0$ (hence the drift constraint is not satisfied) and let $f(x) = 1_{(1,\infty)}(x) + x 1_{(0,1]}(x)$. The process X^* then has the characteristic triplet $(0, \nu, 0)$ (with respect to the same truncation function), and the function $u(x) := E^x[f(X^*_{\tau_1^*})]$ is given by (see [9]),

$$u(x) = 2^{1-\alpha} \Gamma(\alpha) \left[\Gamma\left(\frac{\alpha}{2}\right) \right]^{-2} \int_{-1}^{x} (1 - u^2)^{\alpha/2 - 1} du$$

for $|x| < 1$ and $u(x) = f(x)$ for $|x| \geq 1$. Observe that for $|x| < 1$,

$$u'(x) \geq 2^{1-\alpha} \Gamma(\alpha) \left[\Gamma\left(\frac{\alpha}{2}\right) \right]^{-2} := C$$

and (this is shown in [9])

$$\int_{\mathbb{R}} \left\{ u(x + z) - u(x) - z u'(x) \right\} \nu(dz) = 0.$$

Using this identity in the Itô formula applied to $u(X_t^\varepsilon)$ between $t = 0$ and $t = \tau_\delta^\varepsilon$ for $\delta \in (0, 1)$ (to avoid regularity issues), and taking the expectation, we get

$$E[u(X_{\tau_\delta^\varepsilon}^\varepsilon) - u(0)] = \varepsilon^{\alpha-1} \gamma_c E\left[\int_0^{\tau_\delta^\varepsilon} u'(X_s^\varepsilon) \right] \geq C \varepsilon^{\alpha-1} \gamma_c E[\tau_\delta^\varepsilon],$$

which is equivalent to

$$E[u(\delta X_{\tau_1^{\varepsilon\delta}}^{\varepsilon\delta}) - u(0)] \geq C \varepsilon^{\alpha-1} \delta^\alpha \gamma_c E[\tau_1^{\varepsilon\delta}].$$

With the notation $\rho = \varepsilon\delta$, this gives

$$E[u(\delta X_{\tau_1^\rho}^\rho) - u(0)] \geq C \rho^{\alpha-1} \delta \gamma_c E[\tau_1^\rho].$$

Taking the limit $\delta \to 1$ then yields

$$E[f(X_{\tau_1^\rho}^\rho) - f(X_{\tau_1^*}^*)] = E[u(X_{\tau_1^\rho}^\rho) - u(0)] \geq C \rho^{\alpha-1} \gamma_c E[\tau_1^\rho],$$

which is bounded from below by $\rho^{\alpha-1}$ times a positive constant since $E[\tau_1^\rho]$ converges to $E[\tau_1^*]$.

4 Law of large numbers and statistical applications

In this section we give the law of large numbers for the the processes of the form

$$V^\varepsilon(f)_t = \sum_{T_i^\varepsilon \leq t} f\left(\varepsilon^{-1} (Y_{T_i^\varepsilon} - Y_{T_{i-1}^\varepsilon}) \right),$$

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where f is a bounded continuous function on $\mathbb{R}$. Let

$$m(f) = \frac{E[f(X^*_{\tau^*_1})]}{E[\tau^*_1]}.$$

Theorem 1. *Let X be a Lévy process with characteristic triplet (A, ν, γ), satisfying one of the conditions 1–5 of Proposition 1. Let f be a bounded continuous function on $\mathbb{R}$. Then*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^\alpha V^\varepsilon(f)_t = m(f)S_t \tag{11}$$

in probability, uniformly on compact sets in t (ucp).

As shown in the following examples, this result can be in particular used to build estimators of relevant quantities such as the time change or the Blumenthal-Getoor index.

Example 1 (Estimation of the time change). Assume that the parameters of the underlying Lévy process are known. In our model, the time change can be recovered simply from the times (T^ε_i) as $\varepsilon \to 0$, by taking $f = 1$, which gives,

$$S_t = \lim_{\varepsilon \downarrow 0} \varepsilon^\alpha V^\varepsilon(1)_t E[\tau^*_1]. \tag{12}$$

Example 2 (Estimation of the Blumenthal-Getoor index for the time-changed CGMY process). Let X be the CGMY process (6) with $1 < \alpha < 2$. Including the constant C into the time change, we can assume $C = 1$ with no loss of generality. In this case, the limiting process X^* is a symmetric α-stable process and has Lévy density $\nu^*(x) = \frac{1}{|x|^{1+\alpha}}$. Our method allows therefore to estimate the Blumenthal-Getoor index α of the process X. The coefficients λ_+ and λ_- cannot be identified from the trajectory of the process over a finite time interval, even in the case of continuous observation.

The law of the symmetric stable process at the first exit time from an interval is well known in the literature [9, 19]: $X^*_{\tau^*_1}$ has density

$$\mu(y) = \frac{1}{\pi} \sin\left(\frac{\pi \alpha}{2}\right) |y|^{-1} (y^2 - 1)^{-\frac{\alpha}{2}}, \quad |y| \geq 1.$$

and

$$E[\tau^*_1] = \frac{\sqrt{\pi}}{2^\alpha \Gamma\left(1 + \frac{\alpha}{2}\right)}. \tag{13}$$

With $f(x) = \frac{1}{|x|^\beta} \wedge 1$, $\beta \geq 0$ we easily get

$$E[f(X^*_{\tau^*_1})] = \int_{|y| \geq 1} \frac{\mu(y)}{|y|^\beta} dy = \frac{\Gamma\left(\frac{\alpha}{2} + \frac{\beta}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(1 + \frac{\beta}{2}\right)},$$

where Γ is the gamma function, and in particular for $\beta = 2$, $E[(X^*_{\tau^*_1})^{-2}] = \frac{\alpha}{2}$. Combining (11) and (12), we then obtain a consistent estimator of α:

$$\alpha = 2 \lim_{\varepsilon \downarrow 0} \frac{V^\varepsilon(f)_t}{V^\varepsilon(1)_t}, \quad f(x) = \frac{1}{x^2} \wedge 1.$$

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5 Central limit theorem and convergence rates for estimators

We now turn to the central limit theorem. The following result establishes the rate of convergence and asymptotic normality of the renormalized error in (11).

Theorem 2. *Assume that the time-change S defining Y is independent of the underlying Lévy process X.*

Let X be a Lévy process satisfying Condition 1 of Proposition 3 and let $d \in \mathbb{N}^*$ and $f_1, \ldots, f_d$ be bounded Lipschitz functions on $\mathbb{R}$ satisfying $f_i(1) = f_i(-1)$ for $i = 1, \ldots, d$

or

let X be a Lévy process satisfying Condition 3 of Proposition 3 and let $d \in \mathbb{N}^*$ and $f_1, \ldots, f_d$ be bounded continuous functions on $\mathbb{R}$.

Define $R_t^\varepsilon = (R_{t,1}^\varepsilon, \ldots, R_{t,d}^\varepsilon)$ with

$$R_{t,j}^\varepsilon = \varepsilon^{-\alpha/2}(\varepsilon^\alpha V^\varepsilon(f_j)_t - m(f_j)S_t).$$

Then, as ε goes to zero, R^ε converges in law to $B \circ S$, for the usual Skorohod topology, with B a continuous centered $\mathbb{R}^d$−valued Gaussian process with independent increments, independent of S, such that $E[B_{t,j}B_{t,k}] = (t/(E[\tau_1^*])C_{j,k}$ with

$$C_{j,k} = \mathrm{Cov}[f_j(X_{\tau_1^*}^*) - m(f_j)\tau_1^*, f_k(X_{\tau_1^*}^*) - m(f_k)\tau_1^*].$$

Under Condition 2 of Proposition 3, τ_1^* et $X_{\tau_1^*}^*$ are deterministic, and therefore a central limit theorem cannot be established. In this case, we can only provide an upper bound on the error of the estimators.

Proposition 4. *Let X be a Lévy process satisfying Condition 2 of Proposition 3, and let f be a real bounded Lipschitz function on $\mathbb{R}$. Then, for every $\delta > 0$,*

$$\varepsilon^{-(1-\beta-\delta)\vee -\frac{1}{2}}\{\varepsilon V^\varepsilon(f)_t - m(f)S_t\} \to 0$$

as $\varepsilon \to 0$, in probability uniformly in t on compacts.

6 Proofs

We give in this section the proofs of the preceding results.

6.1 Proof of Proposition 1

Let (A^*, ν^*, γ^*) denote the characteristic triplet of the limiting process. By corollary VII.3.6 in [23], in order to prove the convergence in law, we need to

check that

$$\gamma^\varepsilon \to \gamma^*; \tag{14}$$

$$A^\varepsilon + \int_{\mathbb{R}} h^2(x)\nu^\varepsilon(dx) \to A^* + \int_{\mathbb{R}} h^2(x)\nu^*(dx); \tag{15}$$

$$\text{and} \quad \int_{\mathbb{R}} f(x)\nu^\varepsilon(dx) \to \int_{\mathbb{R}} f(x)\nu^*(dx) \tag{16}$$

for every continuous bounded function f which is zero in a neighborhood of zero.

Part 1   We first check (14). Using the explicit form of the truncation function, we get, for $\varepsilon < 1$,

$$|\gamma^\varepsilon| \le \varepsilon|\gamma| + \varepsilon\int_{|x|>1}\nu(dx) + \varepsilon\int_{\varepsilon<x\le1}(x-\varepsilon)\nu(dx) + \varepsilon\int_{-1\le x<-\varepsilon}(\varepsilon-x)\nu(dx).$$

The convergence of the first two terms to zero is evident; for the third term it is the consequence of the dominated convergence theorem, because the integrand $\varepsilon(x-\varepsilon)1_{\varepsilon<x\le1}$ converges to zero and is bounded from above by $x^2 1_{0<x\le1}$, and the fourth term is treated similarly to the third one. Therefore, $\gamma^\varepsilon \to 0 = \gamma^*$.

To prove (15), we observe that $A^\varepsilon \to A$ and moreover

$$\int_{\mathbb{R}} h^2(x)\nu^\varepsilon(dx) = \int_{|x|\le\varepsilon} x^2\nu(dx) + \varepsilon^2\int_{|x|>1}\nu(dx) + \varepsilon^2\int_{\varepsilon<|x|\le1}\nu(dx).$$

For the first two terms the convergence to zero is evident, and for the last one we can once again apply the dominated convergence theorem using the fact that $\varepsilon^2 1_{\varepsilon<|x|\le1} \le x^2 1_{0<|x|\le1}$.

For the condition (16), assume $f(x) = 0$ for $|x| \le \delta$. Then we can again decompose

$$\int_{\mathbb{R}} f(x)\nu^\varepsilon(dx) = \varepsilon^2\int_{\delta\varepsilon<|x|\le1} f(x/\varepsilon)\nu(dx) + \varepsilon^2\int_{|x|>1} f(x/\varepsilon)\nu(dx),$$

and apply the dominated convergence theorem to the first term, to show that the limit is zero.

Part 2   The proof of this part is a minor modification of part 1, so we omit it to save space.

Conditions (15) and (16) in parts 3, 4 and 5 To prove (15), we fix $\eta > 0$ such that $g(x)$ is bounded on $[-\eta, \eta]$. Then

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} h^2(x) \nu^\varepsilon(dx) = \lim_{\varepsilon \downarrow 0} \varepsilon^\alpha \int_{|x| \le \eta} \frac{h^2(x/\varepsilon) g(x) dx}{|x|^{1+\alpha}}$$

$$= \lim_{\varepsilon \downarrow 0} \int_{|x| \le \eta/\varepsilon} \frac{h^2(x) g(\varepsilon x) dx}{|x|^{1+\alpha}} = \int_{\mathbb{R}} h^2(x) \nu^*(dx),$$

where in the last equality we use the dominated convergence theorem. The condition (16) is shown in a similar manner.

Condition (14) in part 3 Since $\alpha > 1$ and h is bounded, for every $\eta > 0$,

$$\lim_{\varepsilon \downarrow 0} \gamma^\varepsilon = \lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha-1} \int_{|x| \le \eta} \nu(dx) (\varepsilon h(x/\varepsilon) - h(x))$$

Since g has left and right limit at zero, for every $\delta > 0$ we can choose $\eta < 1$ small enough so that $|g(x) - c^+| < \delta$ for $0 < x \le \eta$ and $|g(x) - c^-| < \delta$ for $-\eta \le x < 0$. Then, using the explicit form of h,

$$\lim_{\varepsilon \downarrow 0} \gamma^\varepsilon \le \lim_{\varepsilon \downarrow 0} \left\{ \int_\varepsilon^\eta \frac{(c^+ - \delta)(\varepsilon - x) dx}{|x|^{1+\alpha}} + \int_{-\eta}^{-\varepsilon} \frac{(c^+ + \delta)(-\varepsilon - x) dx}{|x|^{1+\alpha}} \right\}$$

$$\lim_{\varepsilon \downarrow 0} \gamma^\varepsilon \ge \lim_{\varepsilon \downarrow 0} \left\{ \int_\varepsilon^\eta \frac{(c^+ + \delta)(\varepsilon - x) dx}{|x|^{1+\alpha}} + \int_{-\eta}^{-\varepsilon} \frac{(c^+ - \delta)(-\varepsilon - x) dx}{|x|^{1+\alpha}} \right\}$$

Explicit evaluation of these integrals together with the fact that the choice of δ is arbitrary, yields

$$\lim_{\varepsilon \downarrow 0} \gamma^\varepsilon = -\frac{c_+ - c_-}{\alpha(\alpha - 1)},$$

and it is easy to check that the third component of the characteristic triplet of a Lévy process with Lévy density (5) equals $-\frac{c_+ - c_-}{\alpha(\alpha-1)}$ with the truncation function $h(x) = -1 \vee x \wedge 1$ if and only if it equals zero with $h(x) = x$.

Condition (14) in part 4 We rewrite γ^ε as

$$\gamma^\varepsilon = \gamma + \int_0^\infty \frac{g(x) - g(-x)}{x^2} \{\varepsilon h(x/\varepsilon) - h(x)\} dx$$

and apply the dominated convergence, using the fact that $|\varepsilon h(x/\varepsilon) - h(x)| \le h(x)$.

Condition (14) in part 5 Using the fact that the process has zero drift, we get $\gamma^\varepsilon = \varepsilon^\alpha \int_{\mathbb{R}} \nu(dx) h(x/\varepsilon)$, and once again, choosing $\eta > 0$ such that g is bounded on $[-\eta, \eta]$, we get, by dominated convergence:

$$\lim_{\varepsilon \downarrow 0} \gamma^\varepsilon = \lim_{\varepsilon \downarrow 0} \varepsilon^\alpha \int_{|x| \le \eta} \frac{g(x) h(x/\varepsilon) dx}{|x|^{1+\alpha}} = \lim_{\varepsilon \downarrow 0} \int_{|x| \le \eta/\varepsilon} \frac{g(x\varepsilon) h(x) dx}{|x|^{1+\alpha}} = \int_{\mathbb{R}} h(x) \nu^*(dx).$$

6.2   Proof of Proposition 2

Part 1   This will follow if we show that the mapping which to a trajectory $\alpha \in \mathbb{D}$ (space of càdlàg trajectories) associates $(\tau_1^\alpha, \alpha(\tau_1^\alpha))$, with $\tau_1^\alpha := \inf\{t \geq 0 : |\alpha(t)| \geq 1\}$, is continuous in Skorohod topology. We work component by component. We start with the first component and study the continuity of the mapping which to a trajectory $\alpha \in \mathbb{D}$ associates τ_1^α. This in turn follows from Proposition VI.2.11 in [23], provided that we prove that the processes X^ε for every ε and X^* satisfy two regularity properties:

$$\inf\{t \geq 0 : |Z_t| \geq 1\} = \lim_{\delta\downarrow 1} \inf\{t \geq 0 : |Z_t| \geq \delta\} \tag{17}$$

$$\inf\{t \geq 0 : |Z_t| \geq 1\} \leq \inf\{t \geq 0 : |Z_{t-}| \geq 1\} \tag{18}$$

almost surely, where Z stands for X^ε or X^*. From the proof of Lemma 7.10 in [26], it follows that property (17) holds for every Lévy process unless it is of compound Poisson type, which is excluded by the conditions of Proposition 1.

To show Property (18), we introduce $\tau = \inf\{t \geq 0 : |Z_{t-}| \geq 1\}$. Remark that τ is a stopping time as the hitting time of a Borel set by a càglàd adapted process (debut theorem). Property (18) may fail only if the process Z creeps up to the boundary of $[-1, 1]$ and then immediately jumps back inside this domain, which happens only if $|Z_{\tau-}| = 1$ and $\Delta Z_\tau \neq 0$. Introduce the sequence $\tau_n = \inf\{t \geq 0 : |Z_{t-}| \geq 1 - 1/n\}$, which satisfies $\tau_n \leq \tau$. On the set $\{|Z_{\tau-}| = 1\}$ also $\tau_n < \tau$ for all n and it is clear that $\tau_n \to \tau$. If $|Z_{\tau-}| \neq 1$ it means that the level 1 is attained by a jump, and hence $|Z_{\tau-}| < 1$ and $\tau_n = \tau$ as soon as $1 - 1/n > |Z_{\tau-}|$ so that also $\tau_n \to \tau$. Therefore, by Proposition I.7 in [8], on the set $\{|Z_{\tau-}| = 1\}$, $\Delta Z_\tau = 0$.

The continuity of the second component follows from the proof of Proposition 2.12 in [23] (part c.) together with the inequality (18).

Part 2   We will show that the family $(\tau_1^\varepsilon)_{\varepsilon>0}$ has a uniformly bounded exponential moment, which will imply uniform integrability and convergence of $E[(\tau_1^\varepsilon)^k f(X_{\tau_1^\varepsilon}^\varepsilon)]$. We treat separately Conditions 1, 2 and 3–5 of Proposition 1.

Condition 1   Since any jump ΔX_t^ε with $|\Delta X^\varepsilon| \geq 2$ immediately takes the process X^ε out of the domain $(-1, 1)$, the exit time τ_1^ε is dominated by $\tilde{\tau}_1^\varepsilon := \inf\{t > 0 : |\tilde{X}_t^\varepsilon| \geq 1\}$, where the process $\tilde{X}^\varepsilon$ is obtained from X^ε by truncating all jumps greater than 2 in absolute value. The characteristic exponent of $\tilde{X}^\varepsilon$ is

$$\psi_\varepsilon(u) = -\frac{Au^2}{2} + iu\gamma_\varepsilon + \int_{|x|<2} (e^{iux} - 1 - iux)\nu^\varepsilon(dx),$$

where for simplicity we have assumed that the truncation function satisfies $h(x) = x$ for $|x| < 2$. This can be rewritten as

$$\psi_\varepsilon(u) = -\frac{Au^2}{2} + iu\gamma_\varepsilon + \int_{|x|<2\varepsilon} \varepsilon^2 (e^{iux/\varepsilon} - 1 - iux/\varepsilon)\nu(dx) := -\frac{Au^2}{2} + \tilde{\psi}_\varepsilon(u),$$

and it is easily seen that

$$|\tilde{\psi}_\varepsilon(u)| \leq |\gamma_\varepsilon||u| + \frac{|u|^2}{2}e^{2|u|}\int_{|x|<2\varepsilon} x^2\nu(dx), \quad u \in \mathbb{C}.$$

Since $\gamma_\varepsilon \to 0$ as $\varepsilon \to 0$ (see the proof of Proposition 1), we can find $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ and for all $u \in \mathbb{C}$ with $|u| = \frac{1}{2}$, $\frac{2}{A}|\tilde{\psi}_\varepsilon(u)| < \frac{1}{8}$. From this bound we deduce: $\Im\psi_\varepsilon(e^{i\pi/12}/2) \leq 0$ and $\Im\psi_\varepsilon(e^{-i\pi/12}/2) \geq 0$. From the continuity of ψ_ε it follows that there exists $\theta \in \left[-\frac{\pi}{12}, \frac{\pi}{12}\right]$ such that $u^* := e^{i\theta}/2$ satisfies $\Im\psi_\varepsilon(u^*) = 0$ and $\Re\psi_\varepsilon(u^*) \in [-\frac{3A}{12}, -\frac{A}{16}]$.

Consider now the (complex) exponential martingale $M_t^\varepsilon = e^{iu^*\tilde{X}_t^\varepsilon - t\psi_\varepsilon(u^*)}$. Since $|\tilde{X}_{\tilde{\tau}_1^\varepsilon \wedge t}^\varepsilon| \leq 3$, we get that $E[M_{\tilde{\tau}_1^\varepsilon}^\varepsilon] = 1$, and taking the real part,

$$E[e^{-\tilde{\tau}_1^\varepsilon \psi_\varepsilon(u^*)}] \leq \frac{e^{3|u^*|}}{\cos(3|u^*|)} = \frac{e^{3/2}}{\cos(3/2)},$$

which implies

$$E[e^{\frac{A}{16}\tilde{\tau}_1^\varepsilon}] \leq \frac{e^{3/2}}{\cos(3/2)}$$

for all $\varepsilon < \varepsilon_0$.

Condition 2 Without loss of generality let $\gamma_0 > 0$. We use the Lévy-Itô decomposition of X:

$$X_t = \gamma_0 t + \int_0^t \int_{\mathbb{R}} z J(ds \times dz),$$

where J is the jump measure of X, and we denote

$$\tilde{X}_t := \int_0^t \int_{|z|<2\varepsilon} z J(ds \times dz).$$

Since any jump ΔX with $|\Delta X| \geq 2\varepsilon$ immediately takes the process X^ε out of the domain $(-1, 1)$, for every $k > 1$

$$P\left[\tau_1^\varepsilon > \frac{k}{\gamma_0}\right] \leq P\left[\gamma_0 t + \tilde{X}_t \in (-\varepsilon, \varepsilon), \forall t \leq \frac{k\varepsilon}{\gamma_0}\right] \leq P\left[|\tilde{X}_{\frac{k\varepsilon}{\gamma_0}}| > \varepsilon(k-1)\right].$$

Since $\tilde{X}$ has bounded jumps, all its exponential moments are finite, and therefore for all $\alpha > 0, \beta > 0$ and $t > 0$,

$$P\left[|\tilde{X}_t| \geq \alpha\right] \leq e^{-\alpha\beta} E\left[e^{\beta|\tilde{X}_t|}\right] \leq e^{-\alpha\beta}\exp\left(t\int_{|z|<2\varepsilon}(e^{\beta|z|}-1)\nu(dz)\right).$$

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Taking $\alpha = \varepsilon(k-1)$, $\beta = \frac{1}{\varepsilon}$ and $t = \frac{k\varepsilon}{\gamma_0}$ yields

$$P\left[|\tilde{X}_{\frac{k\varepsilon}{\gamma_0}}| > \varepsilon(k-1)\right] \le e^{1-k}\exp\left(\frac{k\varepsilon}{\gamma_0}\int_{|z|<2\varepsilon}(e^{|z|/\varepsilon}-1)\nu(dz)\right)$$

$$\le e^{1-k}\exp\left(\frac{ke^2}{\gamma_0}\int_{|z|<2\varepsilon}|z|\nu(dz)\right).$$

Since X is a finite variation process, $\int_{|z|\le 1}|z|\nu(dz) < \infty$ and $\lim_{\varepsilon\downarrow 0}\int_{|z|<2\varepsilon}|z|\nu(dz)$, which means that there exist $\varepsilon_0 > 0$, and two constants $c > 0$ and $C > 0$ such that for all $\varepsilon \le \varepsilon_0$ and all $k > 1$,

$$P\left[\tau_\varepsilon > \frac{k}{\gamma_0}\right] \le Ce^{-ck},$$

which ensures the uniform integrability.

Conditions 3–5 For $T > 0$, the event $\{\tau_1^\varepsilon > T\}$ occurs only if the process X^ε does not have any jumps greater or equal to 2 in absolute value on $[0, T]$. Therefore,

$$P[\tau_1^\varepsilon > T] \le \exp\{-T\varepsilon^\alpha \nu((-\infty, -2\varepsilon]\cup[2\varepsilon, +\infty))\}.$$

On the other hand,

$$\varepsilon^\alpha\nu((-\infty, -2\varepsilon]\cup[2\varepsilon, +\infty)) = \int_{-\infty}^{-2}\frac{g(\varepsilon x)dx}{|x|^{1+\alpha}} + \int_2^{+\infty}\frac{g(\varepsilon x)dx}{|x|^{1+\alpha}}$$

is uniformly bounded from below because g has right and left limits at zero, at least one of which is positive.

6.3   Proof of Proposition 3

Part 1 We first prove the rate of convergence for the first exit time. Let $u(x) := E^x[\tau_1^*] = \frac{1-x^2}{2A}1_{|x|\le 1}$. Then,

$$E[\tau_1^\varepsilon - \tau_1^*] = E[u(X_{\tau_1^\varepsilon}^\varepsilon) + \tau_1^\varepsilon - u(0)].$$

By Itô formula (whose application can be justified, e.g., by regularizing the function u),

$$u(X_{\tau_1^\varepsilon}^\varepsilon) + \tau_1^\varepsilon - u(0) = \int_0^{\tau_1^\varepsilon}u'(X_{t-}^\varepsilon)dX_t^\varepsilon + \int_0^{\tau_1^\varepsilon}\left(\frac{A}{2}u''(X_t^\varepsilon) + 1\right)dt$$

$$+ \sum_{t\le\tau_1^\varepsilon:\Delta X_t^\varepsilon\ne 0}(u(X_t^\varepsilon) - u(X_{t-}^\varepsilon) - \Delta X_t^\varepsilon u'(X_{t-}^\varepsilon))$$

$$= \int_0^{\tau_1^\varepsilon}u'(X_{t-}^\varepsilon)dX_t^\varepsilon$$

$$+ \sum_{t\le\tau_1^\varepsilon:\Delta X_t^\varepsilon\ne 0}(u(X_t^\varepsilon) - u(X_{t-}^\varepsilon) - \Delta X_t^\varepsilon u'(X_{t-}^\varepsilon)),$$

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where we used the fact that $\frac{A}{2}u'' + 1 = 0$. Taking the expectation, using the boundedness of u and u' and the fact that the jumps of X have finite variation, we get:

$$E[\tau_1^\varepsilon - \tau_1^*] = E\left[-\frac{\varepsilon\gamma_0}{A} \int_0^{\tau_1^\varepsilon} X_t^\varepsilon dt + \int_0^{\tau_1^\varepsilon} \int_\mathbb{R} \{u(X_t^\varepsilon + z) - u(X_t^\varepsilon)\}\nu^\varepsilon(dz)dt \right],$$

where $\gamma_0 = \gamma - \int_\mathbb{R} h(x)\nu(dx)$ is the drift of X. Since the limiting process X^* is continuous in this case, using the Skorokhod representation theorem together with the fact that the convergence in Skorokhod topology implies convergence in the local uniform topology (see Theorem VI.1.17 in [23]), we get,

$$\int_0^{\tau_1^\varepsilon} X_t^\varepsilon dt \to \int_0^{\tau_1^*} X_t^* dt$$

in law as $\varepsilon \to 0$. Since τ_1^ε is uniformly integrable and $|X_t^\varepsilon| \leq 1$ before τ_1^ε, also,

$$\lim_{\varepsilon\downarrow 0} E\left[\int_0^{\tau_1^\varepsilon} X_t^\varepsilon dt \right] = E\left[\int_0^{\tau_1^*} X_t^* dt \right] = 0,$$

because X^* is a Brownian motion which is a symmetric process. For the second term under the expectation, we get:

$$\varepsilon^{-1} E\left[\int_0^{\tau_1^\varepsilon} \int_\mathbb{R} \{u(X_t^\varepsilon + z) - u(X_t^\varepsilon)\}\nu^\varepsilon(dz)dt \right]$$

$$= E\left[\int_0^{\tau_1^\varepsilon} \int_\mathbb{R} \varepsilon\{u(X_t^\varepsilon + z/\varepsilon) - u(X_t^\varepsilon)\}\nu(dz)dt \right]$$

which can be shown to go to zero using the boundedness of u and u'.

To compute the convergence rate of the overshoot, we proceed along the same lines, with the function u now defined by $u(x) = f(x)$ for $|x| \geq 1$ and $u(x) = f(1)$ for $|x| < 1$.

Part 2 Once again, we start with the first exit time. Without loss of generality, assume $\gamma_0 > 0$. In this case, $\tau_1^* = \frac{1}{\gamma_0}$. The process X^ε exits the interval $(-1, 1)$ a.s. in finite time, and we denote by $U \subset \Omega$ the set of trajectories on which it exits through the upper barrier. Then,

$$E[\tau_1^\varepsilon] - E[\tau_1^*] = E[(\tau_1^\varepsilon - 1/\gamma_0)1_U] + E[(\tau_1^\varepsilon - 1/\gamma_0)1_{U^c}]$$

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and we analyze the two terms separately. For the first term,

$$\left| E[(\tau_1^\varepsilon - 1/\gamma_0) 1_U] \right| = \frac{1}{\gamma_0} \left| E\left[\left(X_{\tau_1^\varepsilon}^\varepsilon - 1 - \sum_{t \le \tau_1^\varepsilon} \Delta X_t^\varepsilon \right) 1_U \right] \right|$$

$$\le \frac{1}{\gamma_0} E\left[\sum_{t \le \tau_1^\varepsilon} |\Delta X_t^\varepsilon| \wedge 2 \right] = E[\tau_1^\varepsilon] \int_{\mathbb{R}} (|x| \wedge 2) \nu^\varepsilon(dx)$$

$$= E[\tau_1^\varepsilon] \varepsilon \int_{\mathbb{R}} (|x/\varepsilon| \wedge 2) \nu(dx),$$

where the inequality is due to the fact that on U, $|X_{\tau_1^\varepsilon}^\varepsilon - 1| \le \Delta X_{\tau_1^\varepsilon}^\varepsilon$. Then,

$$\varepsilon \int_{\mathbb{R}} (|x/\varepsilon| \wedge 2) \nu(dx) = 2\varepsilon \int_{|x| > 2\varepsilon} \nu(dx) + \int_{|x| \le 2\varepsilon} |x| \nu(dx)$$

$$\le (2\varepsilon)^{1-\beta} \int_{|x| > 2\varepsilon} (|x|^\beta \wedge 1) \nu(dx) + (2\varepsilon)^{1-\beta} \int_{|x| \le 2\varepsilon} |x|^\beta \nu(dx),$$

from which the result for the first term follows.

To treat the second term, we first estimate the probability of the set U^c. If $\sum_{t \le 2/\gamma_0} |\Delta X_t^\varepsilon| \le 1$ then the process X^ε surely exits from the interval $(-1, 1)$ through the upper barrier before time $2/\gamma_0$. Therefore, by the Markov inequality,

$$P[U^c] \le P\left[\sum_{t \le 2/\gamma_0} |\Delta X_t^\varepsilon| > 1 \right]$$

$$\le P\left[\sum_{t \le 2/\gamma_0} |\Delta X_t^\varepsilon| 1_{|\Delta X_t^\varepsilon| \le 1} > 1 \right] + P\left[\exists t \in [0, 2/\gamma_0] : |\Delta X_t^\varepsilon| > 1 \right]$$

$$\le E\left[\sum_{t \le 2/\gamma_0} |\Delta X_t^\varepsilon| 1_{|\Delta X_t^\varepsilon| \le 1} \right] + 1 - \exp\left(\frac{2}{\gamma_0} \nu^\varepsilon((-\infty, -1) \cup (1, \infty)) \right)$$

$$\le \frac{2}{\gamma_0} \int_{|x| \le 1} x \nu^\varepsilon(dx) + \frac{2}{\gamma_0} \int_{|x| > 1} \nu^\varepsilon(dx)$$

$$= \frac{2}{\gamma_0} \int_{|x| \le \varepsilon} x \nu(dx) + \frac{2\varepsilon}{\gamma_0} \int_{|x| > \varepsilon} \nu(dx) = O(\varepsilon^{1-\beta}).$$

The estimate for $E[(\tau_1^\varepsilon - 1/\gamma_0) 1_{U^c}]$ now follows by Cauchy-Schwarz inequality and Proposition 2.

We now move to the convergence rate for the overshoot. Let $u(x) = f(x)$ for $|x| \ge 1$ and $u(x) = f(-1) + \frac{x+1}{2}(f(1) - f(-1))$ for $|x| < 1$. Applying the Itô formula to $f(X_{\tau_1^\varepsilon}^\varepsilon)$ and taking the expectation, we get

$$E[f(X_{\tau_1^\varepsilon}^\varepsilon) - f(X_{\tau_1^*}^*)] = E[f(X_{\tau_1^\varepsilon}^\varepsilon) - f(1)]$$

$$= \frac{f(1) - f(-1)}{2} \gamma_0 \{ E[\tau_1^\varepsilon] - E[\tau_1^*] \} + E\left[\int_0^{\tau_1^\varepsilon} \int_{\mathbb{R}} \{ f(X_s^\varepsilon + z) - f(X_s^\varepsilon) \} \nu^\varepsilon(dz) ds \right],$$

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from which the result follows using the boundedness and the Lipschitz property of f and the convergence rate of the first exit time obtained above.

Part 3 Again, we start with the exit time.

Step 1. Let $\xi \in (0,1)$ such that $c < g(x) < C$ for two constants c and C with $0 < c < C < \infty$ and all $x : |x| \le \xi$. Let $\bar{g}$ be such that $\bar{g}(x) = g(x)$ for all x with $|x| \le \xi$ and $c < g(x) < C$ for all x, let $\bar{\nu}(dx) := \frac{\bar{g}(x)}{|x|^{1+\alpha}} dx$ and let $\bar{J}$ be a Poisson random measure with intensity $\bar{\nu}(dx) \times dt$ independent from J. We define the processes $\bar{X}$ and $\hat{X}$ by

$$\bar{X}_t := \left(\gamma - \int_{|x|>\xi} h(x)\nu(dx)\right)t + \int_0^t \int_{|x|\le\xi} x\tilde{J}(ds \times dx) + \int_0^t \int_{|x|>\xi} x\bar{J}(ds \times dx),$$

$$\hat{X}_t := \left(\gamma - \int_{|x|>\xi} h(x)\nu(dx)\right)t + \int_0^t \int_{|x|\le\xi} x\tilde{J}(ds \times dx).$$

Let $\bar{X}_t^\varepsilon := \varepsilon^{-1}\bar{X}_{\varepsilon^\alpha t}$, $\hat{X}_t^\varepsilon := \varepsilon^{-1}\hat{X}_{\varepsilon^\alpha t}$ and let $\bar{\tau}_1^\varepsilon$ and $\hat{\tau}_1^\varepsilon$ be the corresponding first exit times. By construction, if $\varepsilon \le \xi/2$, $\tau_1^\varepsilon \le \hat{\tau}_1^\varepsilon$ and if $\tau_1^\varepsilon < \hat{\tau}_1^\varepsilon$ then τ_1^ε is the time of the first jump of J which is greater than ξ in absolute value; the same statement holds if τ_1^ε is replaced with $\bar{\tau}_1^\varepsilon$. Let μ^ε be the law of $\hat{\tau}_1^\varepsilon$. It follows that

$$
\begin{aligned}
\left|E[\bar{\tau}_1^\varepsilon - \tau_1^\varepsilon]\right| &\le E[\hat{\tau}_1^\varepsilon - \tau_1^\varepsilon] + E[\hat{\tau}_1^\varepsilon - \bar{\tau}_1^\varepsilon] \\
&\le \int_0^\infty \mu^\varepsilon(dt)t\left(1 - e^{-t\varepsilon^\alpha \nu(\{x:|x|>\xi\})}\right) \\
&\quad + \int_0^\infty \mu^\varepsilon(dt)t\left(1 - e^{-t\varepsilon^\alpha \bar{\nu}(\{x:|x|>\xi\})}\right) \\
&\le \varepsilon^\alpha(\nu(\{x:|x|>\xi\}) + \bar{\nu}(\{x:|x|>\xi\}))\int_0^\infty t^2\mu^\varepsilon(dt) \\
&= \varepsilon^\alpha(\nu(\{x:|x|>\xi\}) + \bar{\nu}(\{x:|x|>\xi\}))E[(\hat{\tau}_1^\varepsilon)^2].
\end{aligned}
$$

Applying the Proposition 2 to the process $\hat{X}^\varepsilon$, we get that $E[(\hat{\tau}_1^\varepsilon)^2]$ is bounded, and therefore, $\left|E[\bar{\tau}_1^\varepsilon - \tau_1^\varepsilon]\right| = O(\varepsilon^\alpha)$.

Step 2. In view of Step 1, it is sufficient to show that

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-\alpha/2}(E[\bar{\tau}_1^\varepsilon] - E[\tau_1^*]) = 0.$$

Let $\mathbb{P}^\varepsilon$ be the probability measure under which the canonical process, denoted by X, follows the same law as $\bar{X}^\varepsilon$, and $\mathbb{P}^*$ be the probability measure under which X follows the same law as X^* By Theorem 33.2 in [38], the restrictions of $\mathbb{P}^\varepsilon$ and $\mathbb{P}^*$ on every finite interval $[0, T]$ are equivalent with density given by

$$\frac{d\mathbb{P}^\varepsilon}{d\mathbb{P}^*}|_{\mathcal{F}_T} = F_T^\varepsilon = \mathcal{E}(U^\varepsilon)_T, \quad U_T^\varepsilon = \int_0^T \int_{\mathbb{R}} (e^{\phi_\varepsilon(x)} - 1)\tilde{J}^{P^*}(dt \times dx),$$

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where $\tilde{J}^{P^*}$ is the compensated jump measure of X under $\mathbb{P}^*$, $\mathcal{E}$ denotes the Doléans-Dade exponential, and $\phi^\varepsilon(x) := \frac{\bar{g}(\varepsilon x)}{c_+ 1_{x>0} + c_- 1_{x<0}}$.

We denote by τ_1 the first exit time of the canonical process out of the interval $(-1, 1)$ and by E^ε and E^* the expectations under the corresponding probabilities. Let $q \in (1 \vee \alpha/\theta, 2)$ and p such that $\frac{1}{q} + \frac{1}{p} = 1$. Then by the monotone convergence theorem and Hölder's inequality,

$$|E^\varepsilon[\tau_1] - E^*[\tau_1]| = |E^*[\tau_1(F^\varepsilon_{\tau_1} - 1)]| \leq E^*[(\tau_1)^p]^{1/p} E^*[|F^\varepsilon_{\tau_1} - 1|^q]^{1/q}$$

The first factor does not depend on ε and is clearly finite (τ_1^* has an exponential moment). As for the second factor, since $F^\varepsilon_t - 1$ is a $\mathbb{P}^*$-martingale starting from zero, by the Burkholder-Davis-Gundy inequality we get,

$$E^*[|F^\varepsilon_{\tau_1} - 1|^q] \leq CE^*\big[[F^\varepsilon]^{q/2}_{\tau_1}\big] = CE^*\big[\big(\sum_{t \leq \tau_1 : \Delta U^\varepsilon_t \neq 0} (F^\varepsilon_{t-})^2 (\Delta U^\varepsilon_t)^2\big)^{q/2}\big]$$

$$\leq CE^*\big[\sum_{t \leq \tau_1 : \Delta U^\varepsilon_t \neq 0} (F^\varepsilon_{t-})^q (\Delta U^\varepsilon_t)^q\big] = CE^*\big[\int_0^{\tau_1} (F^\varepsilon_t)^q dt\big] \int_{\mathbb{R}} (e^{\phi_\varepsilon(x)} - 1)^q \nu^*(dx).$$

The second factor satisfies

$$\int_{\mathbb{R}} \left(e^{\phi_\varepsilon(x)} - 1\right)^q \nu^*(dx) = \varepsilon^\alpha \int_{\mathbb{R}} \left(e^{\phi_1(x)} - 1\right)^q \nu^*(dx) = O(\varepsilon^\alpha)$$

by the Hölder property of g. For the first factor we get:

$$E^*\left[\int_0^{\tau_1} (F^\varepsilon_t)^q dt\right] \leq E^*[\tau_1] + E^*\left[\int_0^{\tau_1} (F^\varepsilon_t)^2 dt\right]$$

$$= E^*[\tau_1] + \int_0^\infty E^*[(F^\varepsilon_t)^2 1_{t \leq \tau_1}] dt = E^*[\tau_1] + \int_0^\infty E^\varepsilon[F^\varepsilon_t 1_{t \leq \tau_1}] dt$$

To get rid of the stochastic exponential in the last expression, we would like to make another change of probability measure. Since F^ε is not a martingale under $\mathbb{P}^\varepsilon$, we represent it as

$$F^\varepsilon_t = \bar{F}^\varepsilon_t \exp\left(t C_\varepsilon\right),$$

where $\bar{F}^\varepsilon$ is the Doléans-Dade exponential of

$$\bar{U}^\varepsilon_t = \int_0^t \int_{\mathbb{R}} (e^{\phi_\varepsilon(x)} - 1) \tilde{J}^{P^\varepsilon}(dt \times dx),$$

and

$$C_\varepsilon = \int_{\mathbb{R}} \left((e^{\phi_\varepsilon(x)} - 1)\phi_\varepsilon(x) - e^{\phi_\varepsilon(x)} + 1\right) \nu^*(dx)$$

$$= \varepsilon^\alpha \int_{\mathbb{R}} \left((e^{\phi_1(x)} - 1)\phi_1(x) - e^{\phi_1(x)} + 1\right) \nu^*(dx) = O(\varepsilon^\alpha).$$

Then,

$$E^* \left[\int_0^{\tau_1} (F_t^\varepsilon)^q dt \right] \leq E^*[\tau_1] + \bar{E}^\varepsilon[e^{\tau_1 C_\varepsilon}],$$

where $\bar{E}^\varepsilon$ denotes the expectation under the probability $\bar{\mathbb{P}}^\varepsilon$ such that $\frac{d\bar{\mathbb{P}}^\varepsilon}{d\mathbb{P}^\varepsilon}|_{\mathcal{F}_t} = \bar{F}_t^\varepsilon$. Since $C_\varepsilon \to 0$ and $\varepsilon \to 0$ and τ_1 has an exponential moment under $\bar{\mathbb{P}}^\varepsilon$ (the arguments in the proof of Proposition 2), we conclude that the first factor in (6.3) is finite. Combining this with (6.3), the proof is completed.

Let us now turn to the convergence rate for the overshoot. We follow the same steps as above. In step 1, we get, using the boundedness of f,

$$|E[f(\bar{X}_{\hat{\tau}_1^\varepsilon}^\varepsilon)] - E[f(X_{\hat{\tau}_1^\varepsilon}^\varepsilon)]| \leq C\{P[\tau_1^\varepsilon < \hat{\tau}_1^\varepsilon] + P[\bar{\tau}_1^\varepsilon < \hat{\tau}_1^\varepsilon]\} = O(\varepsilon^\alpha).$$

The rest of the proof is carried out in the same way, with some simplifications due to the boundedness of f; for example, the Hölder inequality in (6.3) is not needed.

6.4   Proof of Theorem 1

Introduce an auxiliary sequence of times $(\sigma_i^\varepsilon)_{i \geq 0}$ via $\sigma_0^\varepsilon = 0$ and $\sigma_{i+1}^\varepsilon = \inf\{t > \sigma_i^\varepsilon : |X_t - X_{\sigma_i^\varepsilon}| \geq \varepsilon\}$ for $i \geq 1$. The corresponding counting process is denoted by $M_t^\varepsilon = \sum_{i \geq 1} 1_{\sigma_i \leq t}$, and it clearly satisfies $V^\varepsilon(1)_t = M_{S_t}^\varepsilon$ for all t. We first treat the convergence of the process M_t^ε.

Step 1.   Define the process

$$Z_t^\varepsilon = \sum_{i=1}^{[\varepsilon^{-\alpha}t]} (\sigma_i^\varepsilon - \sigma_{i-1}^\varepsilon),$$

where $[x]$ stands for the integer part of x. We first show that $Z_t^\varepsilon \to tE[\tau_1^*]$ in probability for all t. For every $\Delta > 0$,

$$P\big[|Z_t^\varepsilon - tE[\tau_1^*]| > \Delta\big] \leq P\big[|Z_t^\varepsilon - E[Z_t^\varepsilon]| > \frac{\Delta}{2}\big] + 1_{|E[Z_t^\varepsilon] - tE[\tau_1^*]| > \frac{\Delta}{2}}.$$

The second term converges to zero because $E[Z_t^\varepsilon] = [\varepsilon^{-\alpha}t]E[\sigma_1^\varepsilon] = \varepsilon^\alpha[\varepsilon^{-\alpha}t] \times \varepsilon^{-\alpha}E[\sigma_1^\varepsilon] \to tE[\tau_1^*]$ by Proposition 2. For the second term, Chebyshev's inequality yields:

$$P\big[|Z_t^\varepsilon - E[Z_t^\varepsilon]| > \frac{\Delta}{2}\big] \leq \frac{4\text{Var } Z_t}{\Delta^2} = \frac{4[\varepsilon^{-\alpha}t]\text{Var } \sigma_1^\varepsilon}{\Delta^2} \to 0,$$

because by Proposition 2, $\varepsilon^{-2\alpha}\text{Var } \sigma_1^\varepsilon \to \text{Var } \tau_1^*$ as $\varepsilon \to 0$.

Step 2.   We next show that the convergence takes place uniformly on compact sets in t. Recall first Dini's theorem which states that a deterministic sequence

of nonnegative increasing functions on $\mathbb{R}^+$ converging pointwise to a continuous function also converges locally uniformly. Now we use the fact that proving convergence in probability is equivalent to prove that from any subsequence, one can extract another subsequence converging almost surely. This together with Dini's theorem and the pointwise convergence in Step 1 gives

$$Z_t^\varepsilon \overset{ucp}{\to} tE[\tau_1^*], \text{ as } \varepsilon \to 0.$$

Step 3. Our next objective is to deduce the ucp convergence of M from that of Z. Let $\Delta > 0, T > 0$ and $\bar{M} > T/E[\tau_1^*]$. Since $Z_{M_t^\varepsilon \varepsilon^\alpha}^\varepsilon \leq t$ and $Z_{(1+M_t^\varepsilon)\varepsilon^\alpha}^\varepsilon > t$, we have

$$P[\sup_{t \leq T} |\varepsilon^\alpha M_t^\varepsilon E[\tau_1^*] - t| > \Delta]$$

is smaller than

$$P[\sup_{t \leq T}\{\varepsilon^\alpha M_t^\varepsilon E[\tau_1^*] - Z_{M_t^\varepsilon \varepsilon^\alpha}^\varepsilon\} > \Delta] + P[\sup_{t \leq T}\{Z_{(1+M_t^\varepsilon)\varepsilon^\alpha}^\varepsilon - \varepsilon^\alpha M_t^\varepsilon E[\tau_1^*]\} > \Delta].$$

Thus, for ε small enough, there exists some $c > 0$ such that this is also smaller than

$$2P[M_T^\varepsilon > \bar{M}\varepsilon^{-\alpha}] + 2P\big[\sup_{s \leq \bar{M}+c} |Z_s^\varepsilon - sE[\tau_1^*]| > \Delta - E[\tau_1^*]\varepsilon^\alpha, M_T^\varepsilon \leq \bar{M}\varepsilon^{-\alpha}\big]$$

$$\leq 2P[Z_{\bar{M}}^\varepsilon \leq T] + 2P\big[\sup_{s \leq \bar{M}+c} |Z_s^\varepsilon - sE[\tau_1^*]| > \Delta/2\big].$$

Since $\bar{M} > T/E[\tau_1^*]$, the convergence of $Z_{\bar{M}}^\varepsilon$ to $\bar{M}E[\tau_1^*]$ implies that $P[Z_{\bar{M}}^\varepsilon \leq T]$ goes to zero. This together with the ucp convergence of Z_t^ε in Step 3 gives

$$\varepsilon^\alpha M_t^\varepsilon E[\tau_1^*] \overset{ucp}{\to} t, \text{ as } \varepsilon \to 0.$$

Step 4. Define the process

$$\tilde{Z}_t^\varepsilon(f) = \varepsilon^\alpha \sum_{i=1}^{[\varepsilon^{-\alpha}t]} f\big(\varepsilon^{-1}(X_{\sigma_i^\varepsilon} - X_{\sigma_{i-1}^\varepsilon})\big).$$

As in Step 1, we easily show using Proposition 2 that for $t > 0$,

$$\tilde{Z}_t^\varepsilon(f) \to tE[f(X_{\tau_1^*}^*)],$$

in probability.

Step 5. Following Step 2, we obtain

$$\tilde{Z}_t^\varepsilon(f) \overset{ucp}{\to} tE[f(X_{\tau_1^*}^*)], \text{ as } \varepsilon \to 0$$

applying Dini's theorem separately for the positive and negative parts of f.

Step 6. Let $\Delta > 0$ and $\eta > 0$. Since $\varepsilon^\alpha M_t^\varepsilon E[\tau_1^*]$ tends ucp to t, for big enough ε,

$$P[\sup_{t\leq T} |\varepsilon^\alpha M_t^\varepsilon E[\tau_1^*]E[f(X_{\tau_1^*}^*)] - tE[f(X_{\tau_1^*}^*)]| > \Delta/2] \leq \eta.$$

Thus,

$$P[\sup_{t\leq T} |\tilde{Z}_{\varepsilon^\alpha M_t^\varepsilon}^\varepsilon(f)E[\tau_1^*] - tE[f(X_{\tau_1^*}^*)]| > \Delta]$$

is smaller than

$$P[\sup_{t\leq T} |\tilde{Z}_{\varepsilon^\alpha M_t^\varepsilon}^\varepsilon(f)E[\tau_1^*] - \varepsilon^\alpha M_t^\varepsilon E[\tau_1^*]E[f(X_{\tau_1^*}^*)]| > \Delta/2] + \eta.$$

Following the same lines as in Step 3, we eventually obtain

$$\tilde{Z}_{\varepsilon^\alpha M_t^\varepsilon}^\varepsilon(f) \overset{ucp}{\to} m(f)t, \text{ as } \varepsilon \to 0.$$

Step 7. Finally we write

$$P\big[\sup_{t\leq T} |\varepsilon^\alpha V^\varepsilon(f)_t - m(f)S_t| > \delta\big] = P\big[\sup_{t\leq T} |\tilde{Z}_{\varepsilon^\alpha M_{S_t}^\varepsilon}^\varepsilon(f) - m(f)S_t| > \delta\big]$$

$$\leq P\big[\sup_{t\leq T} |\tilde{Z}_{\varepsilon^\alpha M_{S_t}^\varepsilon}^\varepsilon(f) - m(f)S_t| > \delta, S_T \leq T^*\big] + P[S_T > T^*].$$

Choosing first T^* large enough to make $P[S_T > T^*]$ small, we can then take ε small enough to make the first term small as well. This completes the proof.

6.5 Proof of Theorem 2

In this proof, we assume without loss of generality that the function f_1 is constant such that $f_1(x) = 1$.

Step 1. Let $\bar{R}_t^\varepsilon = (\bar{R}_{t,1}^\varepsilon, \ldots, \bar{R}_{t,d}^\varepsilon)$ be defined by

$$\bar{R}_{t,j}^\varepsilon = \varepsilon^{-\alpha/2}\big(\tilde{Z}_{\varepsilon^\alpha M_t^\varepsilon}^\varepsilon(f_j) - tm(f_j)\big).$$

It is in fact sufficient to show that $\bar{R}^\varepsilon$ tends to B. Indeed, in that case, the sequence $(\bar{R}^\varepsilon, S)$ is $C-$tight (see Corollary VI.3.33 in [23]). Using the independence of S, we obtain the convergence of finite dimensional law and finally the convergence in law of $(\bar{R}^\varepsilon, S)$ to (B, S). Now using Skorohod representation theorem, we can place ourselves on the probability space on which this convergence holds almost surely in Skorohod topology. We conclude using the fact that for x in the d dimensional Skorohod space and y an increasing function the 1 dimensional Skorohod space function, the application $(x, y) \to (x \circ y)$ is continuous at continuous (x, y) in Skorohod topology.

Step 2. In this step we study the convergence of the process $L_t^\varepsilon = (L_{t,1}^\varepsilon, \ldots, L_{t,d}^\varepsilon)$ defined by

$$L_{t,j}^\varepsilon = \varepsilon^{-\alpha/2}\big(\tilde{Z}_{t/E[\tau_1^*]}^\varepsilon(f_j) - m(f_j)Z_{t/E[\tau_1^*]}^\varepsilon\big).$$

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We write
$$L_{t,j}^{\varepsilon} = \sum_{i=1}^{[t/(E[\tau_1^*]\varepsilon^{\alpha})]} \xi_{i,j}^{\varepsilon},$$

with
$$\xi_{i,j}^{\varepsilon} = \varepsilon^{\alpha/2} f_j\big(\varepsilon^{-1}(X_{\sigma_i^{\varepsilon}} - X_{\sigma_{i-1}^{\varepsilon}})\big) - \varepsilon^{-\alpha/2} m(f_j)(\sigma_i^{\varepsilon} - \sigma_{i-1}^{\varepsilon}).$$

Using that
$$\big\{\varepsilon^{-1}(X_{\sigma_i^{\varepsilon}} - X_{\sigma_{i-1}^{\varepsilon}}), \sigma_i^{\varepsilon} - \sigma_{i-1}^{\varepsilon}\big\}$$

and $\{X_{\tau_1^{\varepsilon}}^{\varepsilon}, \varepsilon^{\alpha}\tau_1^{\varepsilon}\}$ have the same law, we get

$$\begin{aligned}
E[\xi_{i,j}^{\varepsilon}] &= \varepsilon^{\alpha/2}\big(E[f_j(X_{\tau_1^{\varepsilon}}^{\varepsilon})] - m(f_j)E[\tau_1^{\varepsilon}]\big) \\
&= \varepsilon^{\alpha/2}\big(E[f_j(X_{\tau_1^{\varepsilon}}^{\varepsilon})] - E[f_j(X_{\tau_1^*}^*)] + m(f_j)(E[\tau_1^*] - E[\tau_1^{\varepsilon}])\big)
\end{aligned}$$

and for $1 \le j, k \le d$,

$$E[\xi_{i,j}^{\varepsilon}\xi_{i,k}^{\varepsilon}] = \varepsilon^{\alpha} E\big[\big(f_j(X_{\tau_1^{\varepsilon}}^{\varepsilon}) - m(f_j)\tau_1^{\varepsilon}\big)\big(f_k(X_{\tau_1^{\varepsilon}}^{\varepsilon}) - m(f_k)\tau_1^{\varepsilon}\big)\big].$$

Moreover, for some positive constant c,

$$E[(\xi_{i,j}^{\varepsilon})^4] \le c\varepsilon^{2\alpha}.$$

From the specific assumptions on X for Theorem 2, we get

$$\sum_{i=1}^{[t/(E[\tau_1^*]\varepsilon^{\alpha})]} E[\xi_{i,j}^{\varepsilon}] \to 0.$$

Now, using Proposition 2, we obtain

$$\sum_{i=1}^{[t/(E[\tau_1^*]\varepsilon^{\alpha})]} \big(E[\xi_{i,j}^{\varepsilon}\xi_{i,k}^{\varepsilon}] - E[\xi_{i,j}^{\varepsilon}]E[\xi_{i,k}^{\varepsilon}]\big) \to (t/E[\tau_1^*])C_{j,k}$$

with
$$C_{j,k} = \mathrm{Cov}[f_j(X_{\tau_1^*}^*) - m(f_j)\tau_1^*, f_k(X_{\tau_1^*}^*) - m(f_k)\tau_1^*].$$

Using a usual theorem on the convergence of triangular arrays, see Theorem VIII.3.32 in [23], we obtain that L^{ε} converges in law to a continuous centered $\mathbb{R}^d$-valued Gaussian process with independent increments B such that $E[B_{t,j}B_{t,k}] = (t/(E[\tau_1^*]))C_{j,k}$.

Step 3. We introduce two families of time changes converging ucp to identity: $\eta_t^{\varepsilon} = \varepsilon^{\alpha} M_t^{\varepsilon} E[\tau_1^*]$ and $\bar{\eta}_t^{\varepsilon} = \varepsilon^{\alpha}(1 + M_t^{\varepsilon})E[\tau_1^*]$. Since the ucp convergence implies the convergence in law in the Skorohod space, the sequences η_t^{ε} and $\bar{\eta}_t^{\varepsilon}$ are C-tight. The sequence L_t^{ε} being also C-tight, the sequence of $d + 2$-dimensional processes $(L_t^{\varepsilon}, \eta_t^{\varepsilon}, \bar{\eta}_t^{\varepsilon})$ is C-tight. Since the time changes converge to deterministic limits, we also get the finite dimensional convergence of the preceding sequence

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which implies its convergence in law in the Skorohod space for the Skorohod topology.

By the Skorohod representation theorem, we can place ourselves on the probability space on which $L^\varepsilon \to B$, $\eta_t^\varepsilon \to t$ and $\bar\eta_t^\varepsilon \to t$ almost surely in Skorohod topology. Using again the continuity of composition by time change at continuous limits, we get that $L_{\eta_t^\varepsilon}^\varepsilon \to B_t$ and $L_{\bar\eta_t^\varepsilon}^\varepsilon \to B_t$. Since B_t is continuous, this implies $L_{\eta_t^\varepsilon}^\varepsilon - L_{\bar\eta_t^\varepsilon}^\varepsilon \to 0$ and so, using that $f_1(x) = 1$,

$$\varepsilon^{\alpha/2} + \varepsilon^{-\alpha/2}\big(m(f_1)(Z_{\varepsilon^\alpha(M_t^\varepsilon+1)}^\varepsilon - Z_{\varepsilon^\alpha M_t^\varepsilon}^\varepsilon)\big) \to 0,$$

which gives

$$\varepsilon^{-\alpha/2}(Z_{\varepsilon^\alpha M_t^\varepsilon}^\varepsilon - Z_{\varepsilon^\alpha(M_t^\varepsilon+1)}^\varepsilon) \to 0.$$

This also implies the convergence for the local uniform topology (see Theorem VI.1.17 in [23]). Since by construction $Z_{M_t^\varepsilon \varepsilon^\alpha}^\varepsilon \le t$ and $Z_{(1+M_t^\varepsilon)\varepsilon^\alpha}^\varepsilon > t$ we get

$$|Z_{\varepsilon^\alpha M_t^\varepsilon}^\varepsilon - t| \le |Z_{\varepsilon^\alpha M_t^\varepsilon}^\varepsilon - Z_{\varepsilon^\alpha(M_t^\varepsilon+1)}^\varepsilon|.$$

Thus,

$$\varepsilon^{-\alpha/2}(Z_{\varepsilon^\alpha M_t^\varepsilon}^\varepsilon - t) \to 0.$$

Eventually, we use that $\bar R_t^\varepsilon = L_{\eta_t^\varepsilon}^\varepsilon + \gamma_t^\varepsilon$, with

$$\gamma_{j,t}^\varepsilon = m(f_j)\varepsilon^{-\alpha/2}(Z_{\varepsilon^\alpha M_t^\varepsilon}^\varepsilon - t).$$

Since $\gamma_t^\varepsilon \to 0$, the result follows.

6.6 Proof of Proposition 4

The idea is to repeat the proof of Theorem 1, using sharper estimates (7) et (8) to obtain the convergence rate. We only give the sketch of the proof.

Define the process

$$U_t^\varepsilon = \varepsilon^{-(1-\delta-\beta)\vee-\frac{1}{2}}\left\{\sum_{i=1}^{[\varepsilon^{-1}t]}(\sigma_i^\varepsilon - \sigma_{i-1}^\varepsilon) - tE[\tau_1^*]\right\}.$$

We recall that in our setting τ_1^* is deterministic, but we stick to the notation of the proof of Theorem 1. Then,

$$U_t^\varepsilon = \varepsilon^{-(1-\delta-\beta)\vee-\frac{1}{2}}\left\{\sum_{i=1}^{[\varepsilon^{-1}t]}(\sigma_i^\varepsilon - \sigma_{i-1}^\varepsilon) - [\varepsilon^{-1}t]E[\sigma_1^\varepsilon]\right\}$$
$$+\varepsilon^{-(1-\delta-\beta)\vee-\frac{1}{2}}\left\{\varepsilon[\varepsilon^{-1}t]E[\tau_1^\varepsilon] - tE[\tau_1^*]\right\}.$$

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The bound (7) implies that the terms in the second line converge to zero uniformly in t on compacts. The terms in the first line, by Kolmogorov's inequality, satisfy

$$P\left[\sup_{t\leq t_0}\varepsilon^{-(1-\delta-\beta)\vee-\frac{1}{2}}\left|\sum_{i=1}^{[\varepsilon^{-1}t]}(\sigma_i^\varepsilon-\sigma_{i-1}^\varepsilon)-[\varepsilon^{-1}t]E[\sigma_1^\varepsilon]\right|\geq\lambda\right]$$

$$\leq\frac{1}{\lambda^2}\varepsilon^{-2(1-\delta-\beta)\vee-1}[\varepsilon^{-1}t_0]\operatorname{Var}\sigma_1^\varepsilon\leq\frac{t_0}{\lambda^2}\operatorname{Var}\tau_1^\varepsilon,$$

which converges to zero as $\varepsilon\to 0$ because $E[\tau_1^\varepsilon]\to E[\tau_1^*]$, $E[(\tau_1^\varepsilon)^2]\to E[(\tau_1^*)^2]$ and τ_1^* is deterministic. We have therefore shown that $U^\varepsilon\xrightarrow{ucp}0$.

Now we repeat the arguments of step 3 of the proof of Theorem 1 to show that

$$\varepsilon^{-(1-\delta-\beta)\vee-\frac{1}{2}}\{\varepsilon M_t^\varepsilon E[\tau_1^*]-t\}\xrightarrow{ucp}0.$$

Finally, we define

$$\tilde{U}_t^\varepsilon(f)=\varepsilon^{-(1-\delta-\beta)\vee-\frac{1}{2}}\left\{\sum_{i=1}^{[\varepsilon^{-1}t]}(f(\varepsilon^{-1}(X_{\sigma_i^\varepsilon}-X_{\sigma_{i-1}^\varepsilon}))-tE[f(X_{\tau_1^*}^*)]\right\}.$$

and show that $\tilde{U}^\varepsilon(f)\xrightarrow{ucp}0$ using the same argument as above. The proof can then be completed by repeating the steps 5–7 of the proof of Theorem 1 with the process $\tilde{Z}^\varepsilon(f)$ replaced by $\tilde{U}^\varepsilon(f)$.

References

[1] Y. AÏT-SAHALIA AND J. JACOD, *Volatility estimators for discretely sampled Lévy processes*, Ann. Stat., 35 (2007), pp. 355–392.

[2] ――, *Estimating the degree of activity of jumps in high frequency data*, Ann. Stat., 37 (2009), pp. 2202–2244.

[3] ――, *Is Brownian motion necessary to model high-frequency data?*, Ann. Stat., forthcoming (2009).

[4] ――, *Testing for jumps in a discretely observed process*, Ann. Stat., 37 (2009), pp. 184–222.

[5] O. BARNDORFF-NIELSEN, *Processes of normal inverse Gaussian type*, Finance Stoch., 2 (1998), pp. 41–68.

[6] O. BARNDORFF-NIELSEN, P. HANSEN, A. LUNDE, AND N. SHEPHARD, *Designing realized kernels to measure the ex post variation of equity prices in the presence of noise*, Econometrica, 76 (2008), pp. 1481–1536.

[7] O. Barndorff-Nielsen, N. Shephard, and M. Winkel, *Limit theorems for multipower variation in the presence of jumps*, Stoch. Proc. Appl., 116 (2006), pp. 796–806.

[8] J. Bertoin, *Lévy Processes*, Cambridge University Press, Cambridge, 1996.

[9] R. M. Blumenthal, R. K. Getoor, and D. B. Ray, *On the distribution of first hits for symmetric stable processes*, T. Am. Math. Soc., 99 (1961), pp. 540–554.

[10] P. Carr, H. Geman, D. Madan, and M. Yor, *The fine structure of asset returns: An empirical investigation*, J. Bus., 75 (2002), pp. 305–332.

[11] P. Carr, H. Geman, D. Madan, and M. Yor, *Stochastic volatility for Lévy processes*, Math. Finance, 13 (2003), pp. 345–382.

[12] D. A. Darling and A. J. F. Siegert, *The first passage problem for a continuous Markov process*, Ann. Math. Stat., 24 (1953), pp. 624–639.

[13] M. Domine, *Moments of the first-passage time of a wiener process with drift between two elastic barriers*, J. Appl. Probab., (1995), pp. 1007–1013.

[14] R. A. Doney and R. A. Maller, *Stability of the overshoot for Lévy processes*, Ann. Appl. Probab., (2002), pp. 188–212.

[15] J. E. Figueroa-Lopez, *Nonparametric estimation of time-changed Lévy models under high-frequency data*, Adv. Appl. Probab., 41 (2009), pp. 1161–1188.

[16] ——, *Central limit theorems for the non-parametric estimation of time-changed Lévy models.* preprint, 2010.

[17] M. Fukasawa, *Discretization error of stochastic integrals.* preprint, 2009.

[18] M. Fukasawa, *Realized volatility with stochastic sampling*, Stoch. Proc. Appl., 120 (2010), pp. 829–852.

[19] R. K. Getoor, *First passage times for symmetric stable processes in space*, T. Am. Math. Soc., 101 (1961), pp. 75–90.

[20] T. Hayashi, J. Jacod, and N. Yoshida, *Irregular sampling and central limit theorems for power variations: the continuous case.* Working paper, 2008.

[21] J. Jacod, *Asymptotic properties of realized power variations and related functionals of semimartingales*, Stoch. Proc. Appl., 118 (2008), pp. 517–559.

[22] J. Jacod, Y. Li, P. Mykland, M. Podolskij, and M. Vetter, *Microstructure noise in the continuous case: the pre-averaging approach*, Stoch. Proc. Appl., 119 (2009), pp. 2249–2276.

[23] J. JACOD AND A. N. SHIRYAEV, *Limit Theorems for Stochastic Processes*, Springer, Berlin, 2nd ed., 2003.

[24] A. KOHATSU-HIGA AND P. TANKOV, *Jump-adapted discretization schemes for Lévy-driven SDEs*, Stoch. Proc. Appl., to appear (2010).

[25] S. KOU, *A jump-diffusion model for option pricing*, Manage. Sci., 48 (2002), pp. 1086–1101.

[26] A. KYPRIANOU, *Introductory Lectures on Fluctuations of Lévy Processes with Applications*, Springer, 2006.

[27] Y. LI, P. MYKLAND, E. RENAULT, L. ZHANG, AND X. ZHENG, *Realized volatility when sampling times are possibly endogenous*. Working paper, 2009.

[28] D. MADAN, P. CARR, AND E. CHANG, *The variance gamma process and option pricing*, European Finance Review, 2 (1998), pp. 79–105.

[29] C. MANCINI, *Non-parametric threshold estimation for models with stochastic diffusion coefficient and jumps*, Scandinavian Journal of Statistics, 36 (2009), pp. 270–296.

[30] R. MERTON, *Option pricing when underlying stock returns are discontinuous*, J. Financ. Econ., 3 (1976), pp. 125–144.

[31] D. REVUZ AND M. YOR, *Continuous martingales and Brownian motion*, Springer Verlag, 1999.

[32] C. ROBERT AND M. ROSENBAUM, *Volatility and covariation estimation when microstructure noise and trading times are endogenous*, Math. Finance, to appear (2009).

[33] ———, *On the Microstructural Hedging Error*, SIAM J. Fin. Math., 1 (2010), pp. 427–453.

[34] B. A. ROGOZIN, *The distribution of the first hit for stable and asymptotically stable walks on an interval*, Theor. Probab. Appl., 17 (1972), pp. 332–338.

[35] M. ROSENBAUM, *Integrated volatility and round-off error*, Bernoulli, 15 (2009), pp. 687–720.

[36] J. ROSIŃSKI, *Tempering stable processes*, Stoch. Proc. Appl., 117 (2007), pp. 677–707.

[37] S. RUBENTHALER, *Numerical simulation of the solution of a stochastic differential equation driven by a Lévy process*, Stoch. Proc. Appl., 103 (2003), pp. 311–349.

[38] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge, UK, 1999.

[39] J. Woerner, *Inference in Lévy-type stochastic volatility models*, Adv. Appl. Probab., 39 (2007), pp. 531–549.

[40] L. Zhang, P. Mykland, and Y. Aït-Sahalia, *A tale of two time scale: determining integrated volatility with noisy high-frequency data*, J. Amer. Stat. Assoc., 100 (2005), pp. 1394–1411.